Linear Programming

This activity is about solving problems involving constraints that can be written as linear inequalities. The aim is usually to maximise or minimise something like profit or costs. If there are only two decision variables, then the problem can be solved using a graph.

Example
A company manufactures two types of stair carpet: standard quality and heavy duty. The company must make at least 150 metres of heavy duty carpet per month to fulfil a standing order, but there is no standing order for standard quality carpet.

They can generally sell all of the carpet they make at a profit of £5 per metre for the standard quality and £8 per metre for heavy duty.

There are two production processes involved in making each type of carpet. The time taken for each metre of each type of carpet and the total factory time available for these processes are given in the table.

	Process A	Process B
Standard Quality	0.1 hour	0.4 hour
Heavy Duty	0.3 hour	0.2 hour
Total	150 hours	200 hours

The company wishes to know how much of each type of carpet it should make to maximise its profit.

How to solve this:
Suppose x metres represents the length of standard carpet and y metres the length of heavy duty carpet that the company produces.
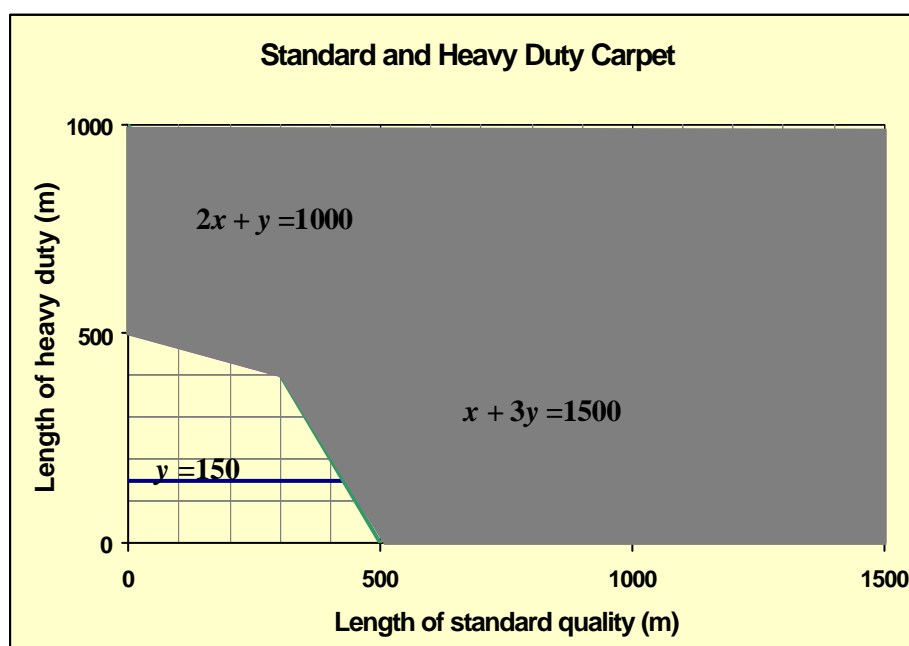The company wishes to maximise the profit £P where $P = 5x + 8y$

The rest of the information gives the following inequalities:
$$y \geq 150 \qquad\qquad 0.1x + 0.3y \leq 150 \qquad 0.4x + 0.2y \leq 200$$

The last two inequalities can be simplified to $\qquad x + 3y \leq 1500 \qquad\qquad 2x + y \leq 1000$
The **unshaded** part of the graph below shows where all three inequalities are satisfied.

Standard and Heavy Duty Carpet

In this case the part of the graph that does not satisfy each inequality was shaded.

This makes the region that satisfies all three inequalities easier to see.

Any point in the unshaded region or on any of its boundaries satisfies all the inequalities.
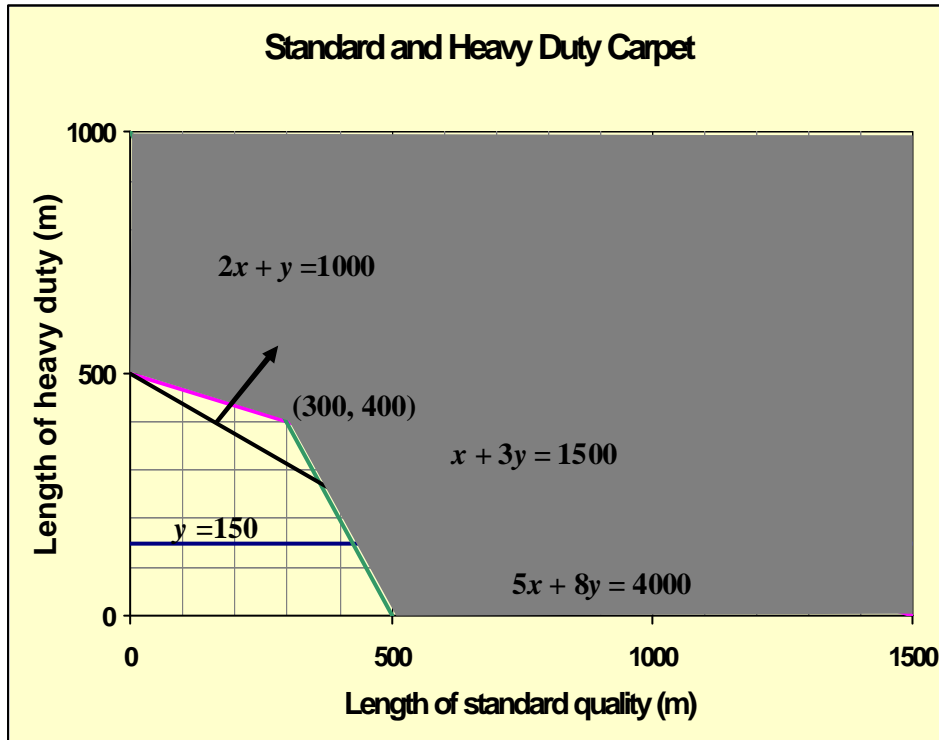
To maximise the profit $P = 5x + 8y$

Different values of P give different equations and different lines on the graph, but they all have the same gradient.

First find the position of one of these lines using a chosen value of P.
The graph below shows the line $5x + 8y = 4000$.
All values of x and y on this line would give a profit of £4000.

Standard and Heavy Duty Carpet

Length of heavy duty (m)

$2x + y = 1000$

500

(300, 400)

$x + 3y = 1500$

$y = 150$

$5x + 8y = 4000$

Length of standard quality (m)

Other values of P would give lines parallel to this one.
The higher the value of P, the further the profit line would be away from the origin.

Imagine the profit line moving away from the origin.
The last point in the clear region that the profit line would pass through is (300, 400) - this gives the solution to the problem. You can check these values by solving the simultaneous equations $x + 3y = 1500$ and $2x + y = 1000$.

To maximise the profit the company should make **300 metres of standard quality carpet** and **400 metres of heavy duty carpet**.

The profit it will make is $P = 5x + 8y = 5 \times 300 + 8 \times 400$

The maximum possible profit = **£4700**

Note
Linear programming is a useful technique, but has limitations:
- Sometimes variables may take only integer values (eg you can only build a whole number of houses). The vertices of the clear region may not have integer coordinates.
- The assumption of linearity is often only valid over a small range of values.
- In many situations it may be difficult to estimate accurately the values used in each algebraic expression.

Other Problems to Solve:

Part-time Jobs

A student earns money by washing cars and gardening.
He has a total of 12 hours available for this work each week.
For cleaning a car he charges £2 and this takes him 20 minutes.
He has six regular customers for whom he washes cars.
He also spends three hours each week mowing lawns for regular
customers and he charges £5 per hour for this and other gardening jobs.

Show that this information gives the inequalities: $\qquad x + y \leq 12$
$$x \geq 2$$
$$y \geq 3$$

where x is the number of hours per week he spends on washing cars and y is the number of
hours he spends gardening.

Use a graph to show all the possible values of x and y that satisfy these inequalities.

Write down, in terms of x and y, an expression for his weekly earnings, £P.

Use your graph to find how much time he should spend on cars and gardening to maximise
his earnings and give the maximum amount he can earn.

Diet Supplements

A study of multi-vitamin tablets includes a comparison
of some of the other ingredients the tablets contain.
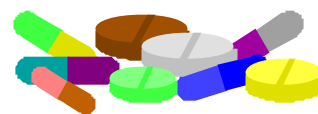
The table gives the results for iron, magnesium
and calcium in Mult-Vit and Vitab tablets.
The last row of the table gives the recommended
daily amount (RDA) of each of these ingredients.

	Iron	Magnesium	Calcium
Mult-Vit	6 mg	200 mg	60 mg
Vitab	1.5 mg	100 mg	80 mg
RDA	18 mg	1000 mg	400 mg

Using x to represent the number of Mult-Vit tablets per day and y the number of Vitab tablets
per day, show that in order to give at least the recommended daily amount of iron, magnesium
and calcium:
$$4x + y \geq 12, \quad 2x + y \geq 10 \quad \text{and} \quad 3x + 4y \geq 20$$

Use a graph to show all the possible values of x and y that satisfy these inequalities.

A tub of 100 Mult-Vit tablets costs £4.20 and a tub of 180 Vitab tablets costs £9
Write down an expression for the cost C pence of x Mult-Vit tablets and y Vitab tablets.

Use your graph to find the number of Mult-Vit and Vitab tablets that would give the
recommended daily dose of iron, magnesium and calcium at the minimum cost per day.
Find also this minimum cost per day.

Teacher Notes

Unit   Advanced Level, Applying mathematics

Notes
Although linear programming is not specifically mentioned in the Applying mathematics specification, it provides a good way of giving practice with linear inequalities in real-life contexts. It is recommended that the resources called 'Linear Inequalities' are used first to introduce the topic.

The example on pages 1 and 2 is used in the accompanying powerpoint presentation which can be used to illustrate the method. Page 3 includes two problems that students can try. Many more examples of linear programming can be found in a variety of A level Decision Mathematics textbooks and examination papers.

Answers

Part-time Jobs

$P = 6x + 5y$
Maximum earnings from 9 hours washing cars and 3 hours gardening = £69

Diet Supplements

$C = 4.2x + 5y$
Minimum cost using 4 Mult-Vit tablets and 2 Vitab tablets = 26.8 pence per day